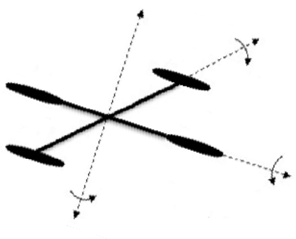
컨볼루션 (Convolution)



- 컨볼루션 적분 (Convolution Intergral)

$$f(t)*g(t)=\int_0^t f(\tau)g(t-\tau)d\tau$$

- Ex

$$e^t*\sin t$$

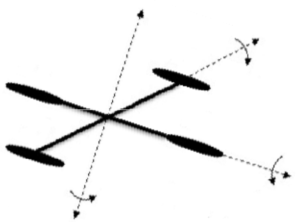
$$=\int_0^t e^\tau \sin(t-\tau)d\tau$$

$$=\left[e^\tau \sin(t-\tau)\right]_0^t+\int_0^t e^\tau \cos(t-\tau)d\tau$$

$$=-\sin t+\left[e^\tau \cos(t-\tau)\right]_0^t-\int_0^t e^\tau \sin(t-\tau)d\tau$$

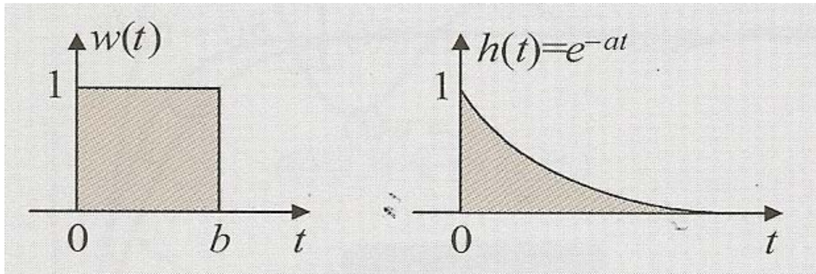
$$2\int_0^t e^\tau \sin(t-\tau)d\tau=-\sin t+e^t-\cos t$$

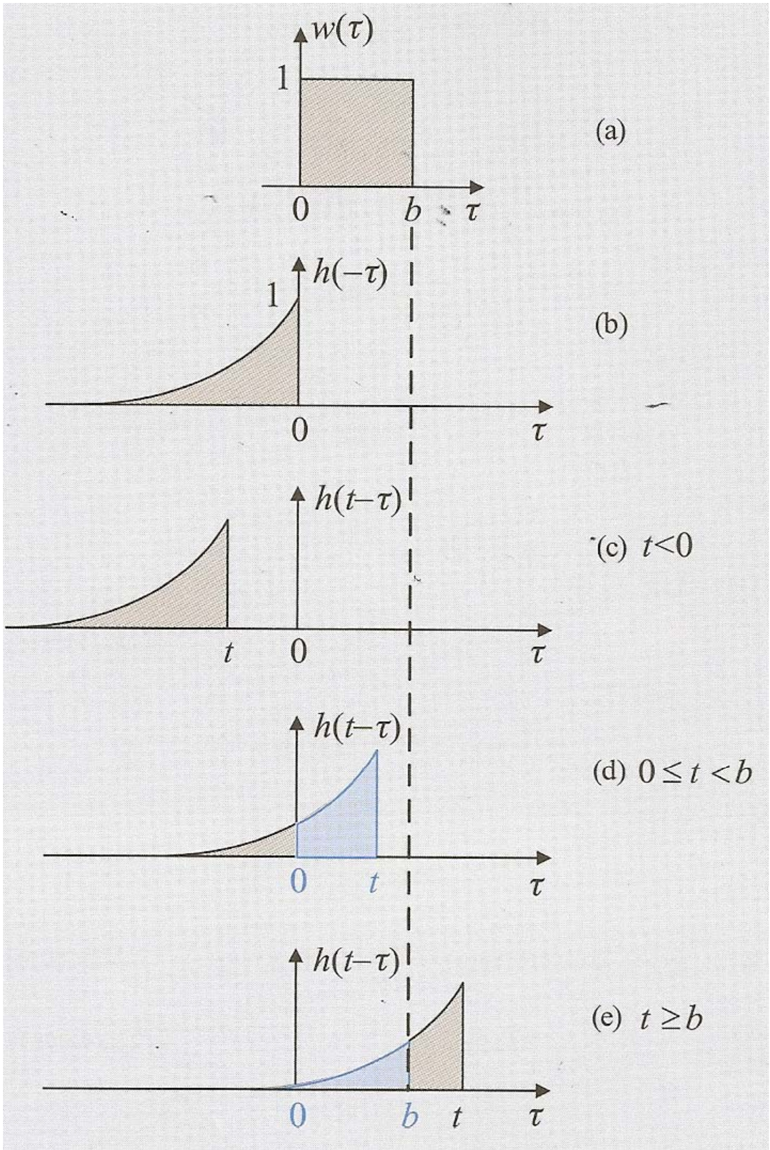
$$e^t*\sin t=\int_0^t e^\tau \sin(t-\tau)d\tau=\frac{1}{2}\left(e^t-\sin t-\cos t\right)$$



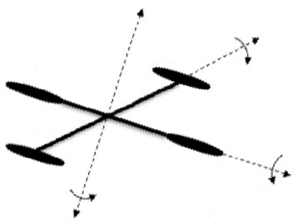
- Ex

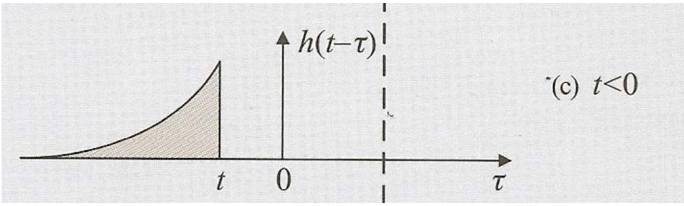
$$y(t) = w(t) * h(t) = \int_{-\infty}^{\infty} w(\tau)h(t-\tau)d\tau$$



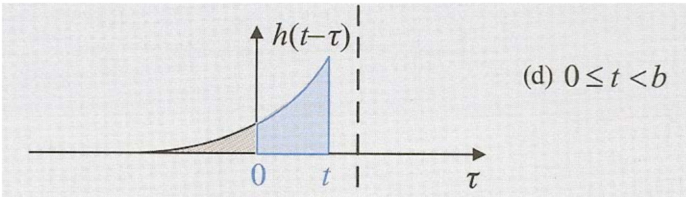


w(τ)
1
0
b
τ
(a)
h(−τ)
1
0
τ
(b)
h(t−τ)
t
0
τ
(c) t<0
h(t−τ)
0
t
τ
(d) 0 ≤ t < b
h(t−τ)
0
b
t
τ
(e) t ≥ b





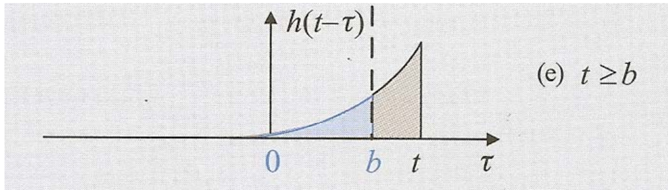
$$y(t) = \int_{-\infty}^{\infty} w(\tau)h(t-\tau)d\tau = 0$$



$$y(t) = \int_{0}^{t} w(\tau)h(t-\tau)d\tau$$

$$= \int_{0}^{t} e^{-a(t-\tau)} \cdot 1 d\tau$$

$$= \frac{1}{a}\left[1 - e^{-at}\right]$$

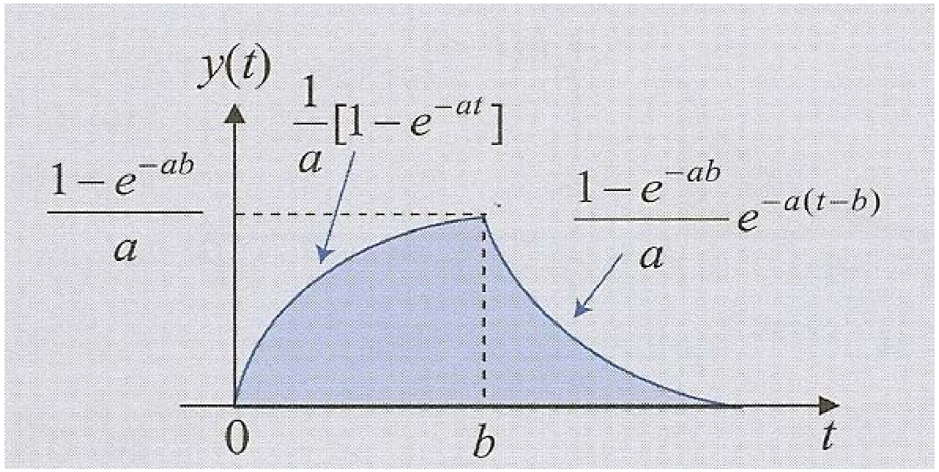


$$y(t) = \int_0^b w(\tau)h(t-\tau)d\tau$$

$$= \int_0^b e^{-a(t-\tau)} \cdot 1 d\tau$$

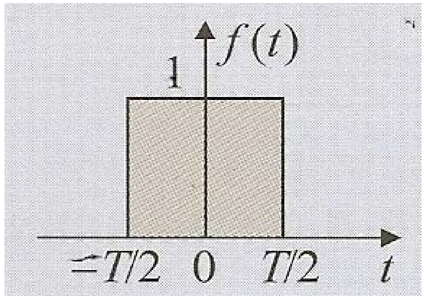
$$= \frac{1-e^{-ab}}{a} e^{-a(t-b)}$$

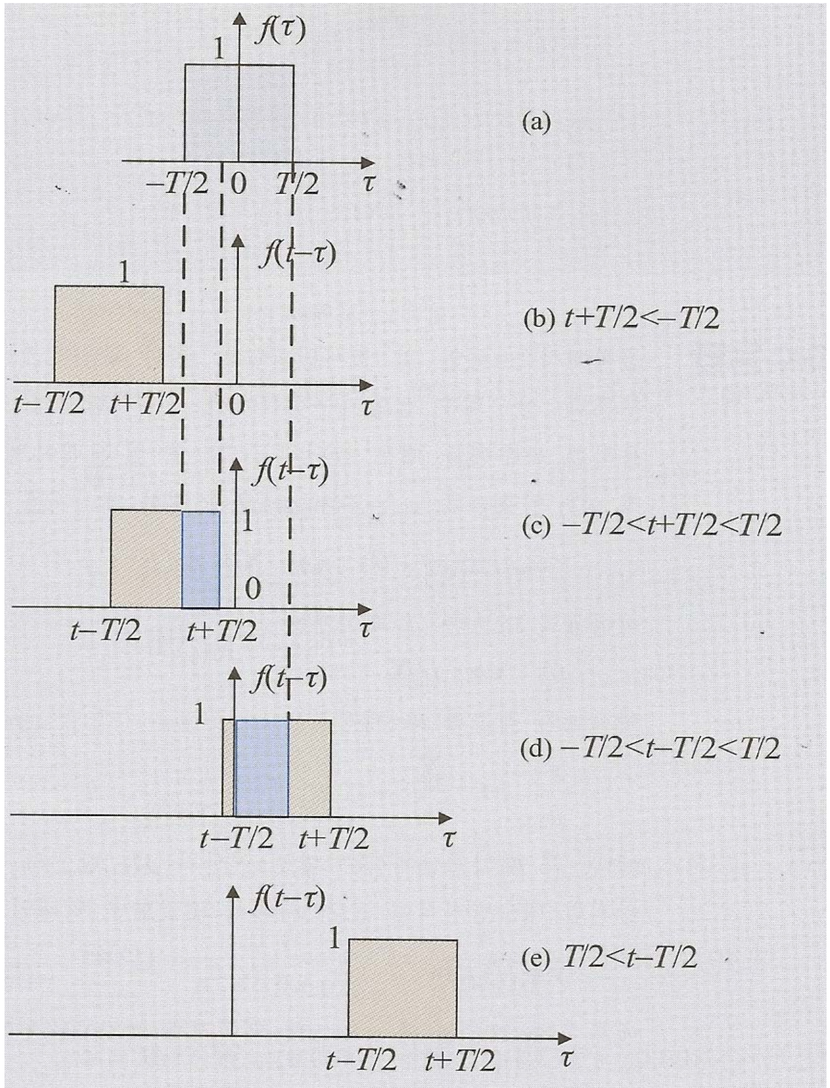
$$y(t) = \begin{cases} 0 & , \ (t<0) \\ \frac{1}{a}[1-e^{-at}] & (0 \le t < b) \\ \frac{1-e^{-ab}}{a} e^{-a(t-b)} & (t \ge b) \end{cases}$$



- Ex

$$y(t) = f(t) * f(t) = \int_{-\infty}^{\infty} f(\tau) f(t-\tau) d\tau$$



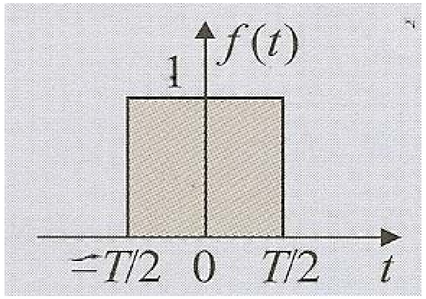


$$f(t) * f(t) = 0$$

$$f(t) * f(t) = \int_{-T/2}^{t+T/2} 1\, d\tau = t + T$$

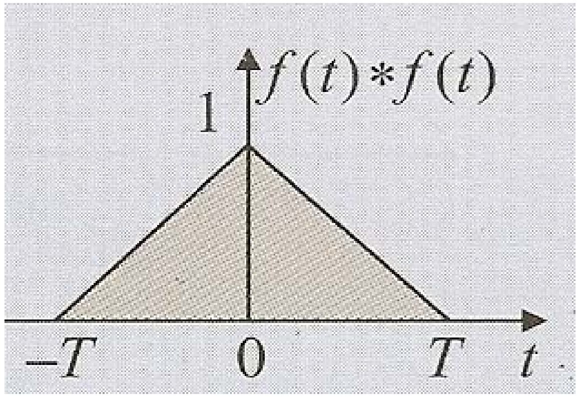
$$f(t) * f(t) = \int_{t-T/2}^{T/2} 1\, d\tau = T - t$$

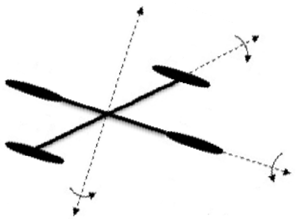
$$f(t) * f(t) = 0$$



$$f(t)*f(t)=\int_{-T/2}^{t+T/2}1\,d\tau=t+T$$

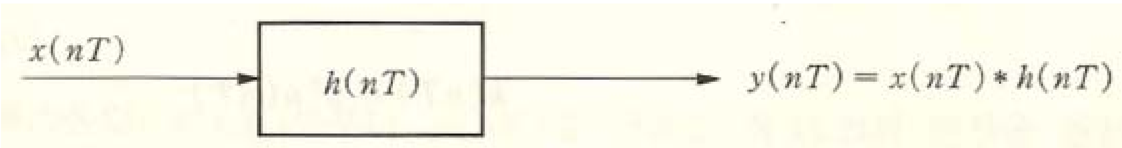
$$f(t)*f(t)=\int_{t-T/2}^{T/2}1\,d\tau=T-t$$

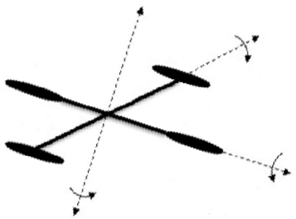




- 이산 시스템에서의 컨볼루션

$$y(nT) = \sum_{k=-\infty}^{\infty} h(kT)x(nT-kT)$$





- Ex

$$x(nT) = h(nT) = \begin{cases} 1, & n = 0,\ 1,\ 2 \\ 0, & \text{기타} \end{cases}$$

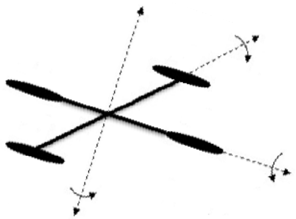
$$y(nT) = \sum_{k=0}^{2} h(kT)x(nT-kT)$$

① $n = 0$일 경우

$$y(0) = h(0)x(0) + h(T)x(-T) + h(2T)x(-2T) = 1$$

② $n = 1$일 경우

$$y(T) = h(0)x(T) + h(T)x(0) + h(2T)x(-T) = 2$$



③ $n = 2$일 경우

$$y(2T) = h(0)\,x(2T) + h(T)x(T) + h(2T)x(0) = 3$$

④ $n = 3$일 경우

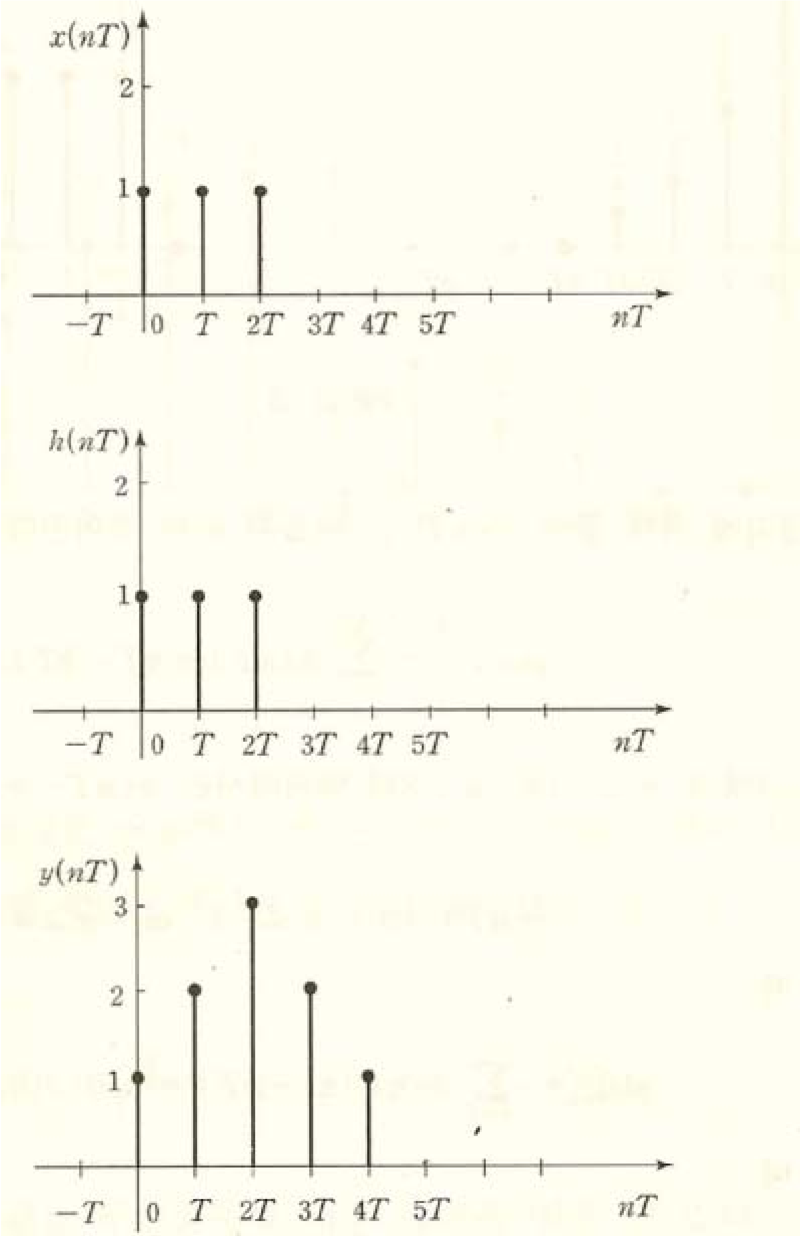
$$y(3T) = h(0)\,x(3T) + h(T)x(2T) + h(2T)x(T) = 2$$

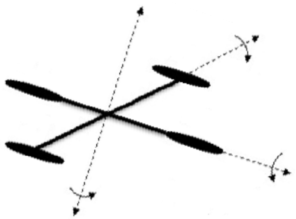
⑤ $n = 4$일 경우

$$y(4T) = h(0)\,x(4T) + h(T)x(3T) + h(2T)x(2T) = 1$$

⑥ $n < 0$과 $n \geq 5$일 경우, 입력신호의 값이 항상 0인 것에 유의하면

$$y(nT) = h(0)\,x(nT) + h(T)x(nT - T) + h(2T)x(nT - 2T) = 0$$

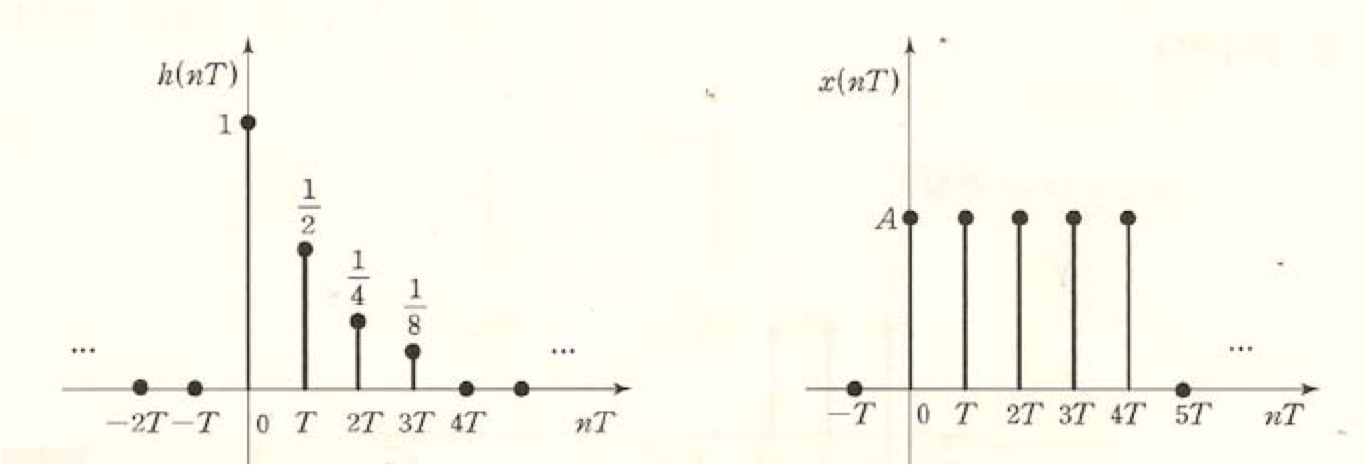

x(nT)
2
1
−T
0
T
2T
3T
4T
5T
nT
h(nT)
2
1
−T
0
T
2T
3T
4T
5T
nT
y(nT)
3
2
1
−T
0
T
2T
3T
4T
5T
nT



- Ex

$$h(nT) = \begin{cases} \left(\frac{1}{2}\right)^n, & 0 \le n \le 3 \\ 0 \quad , & \text{기타} \end{cases}$$

$$x(nT) = \begin{cases} A, & 0 \le n \le 4 \\ 0, & \text{기타} \end{cases}$$



$$y(nT) = \sum_{k=0}^{3} h(kT)x(nT-kT)$$

① 먼저 위의 식에서 $n \leq -1$과 $n \geq 8$의 범위에서는 $x(nT-kT)$의 값은 항상 0이다. 따라서,

$$y(nT)=0, \quad n \leq -1 \quad \text{or} \quad n \geq 8$$

② $n=0$일 때

$$y(0) = \sum_{k=0}^{3} h(kT)x(-kT) = h(0)x(0) = A$$

③ $n=1$일 때

$$y(T) = \sum_{k=0}^{3} h(kT)x(T-kT) = A + \frac{A}{2} = \frac{3}{2}A$$

④ $n = 2$일 때

$$y(2T) = \sum_{k=0}^{3} h(kT)x(2T - kT) = A + \frac{A}{2} + \frac{A}{4} = \frac{7}{4}A$$

같은 방법으로

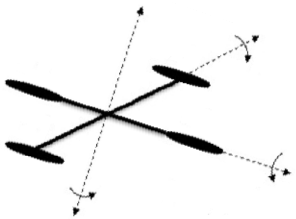
$$y(3T) = A + \frac{A}{2} + \frac{A}{4} + \frac{A}{8} = \frac{15}{8}A$$

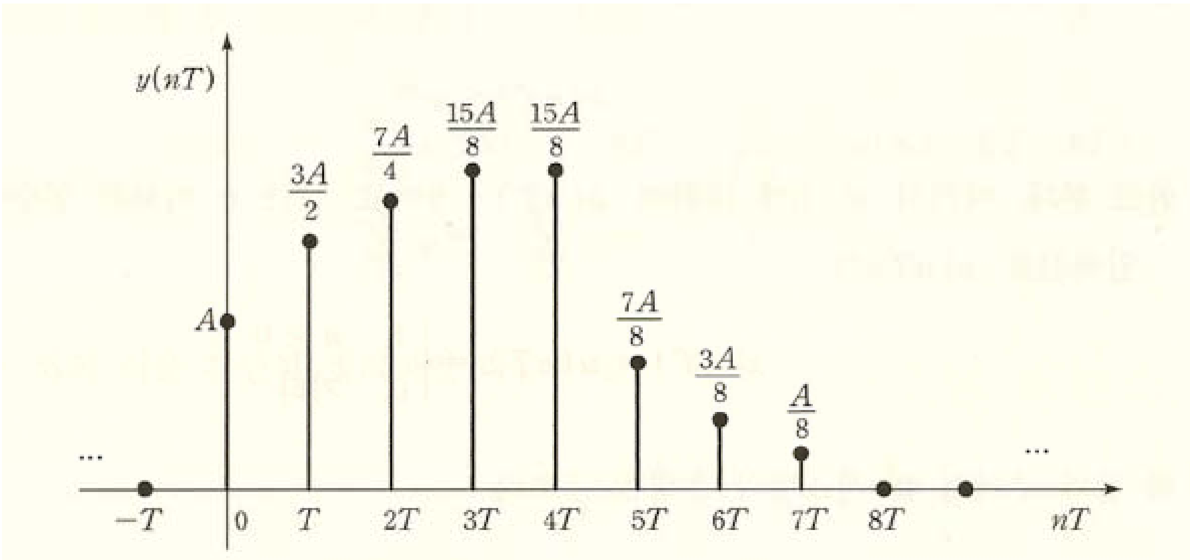
$$y(4T) = A + \frac{A}{2} + \frac{A}{4} + \frac{A}{8} = \frac{15}{8}A$$

$$y(5T) = \frac{A}{2} + \frac{A}{4} + \frac{A}{8} = \frac{7}{8}A$$

$$y(6T) = \frac{A}{4} + \frac{A}{8} = \frac{3}{8}A$$

$$y(7T) = \frac{A}{8}$$





y(nT)
A
3A/2
7A/4
15A/8
15A/8
7A/8
3A/8
A/8
...
...
−T
0
T
2T
3T
4T
5T
6T
7T
8T
nT

예제 3.4 선형 시불변시스템의 임펄스응답이

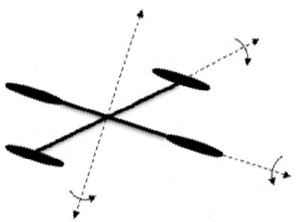
$$h(nT) = e^{n\alpha}$$

라고 한다. 여기서 $n < 0$에 대하여 $h(nT) = 0$이고, α는 0 이외의 실수이다.

입력신호 $x(nT)$가

$$x(nT) = u(nT) = \begin{cases} 1, & n \geq 0 \\ 0, & \text{기타} \end{cases}$$

와 같이 주어질 때 시스템의 출력을 구하라.



$$y(nT) = \sum_{k=0}^{\infty} e^{ka} u(nT - kT)$$

$$\begin{aligned} y(nT) &= e^0 u(nT) + e^a u(nT - T) + e^{2a} u(nT - 2T) + \cdots \\ &\quad + e^{na} u(0) + e^{(n+1)a} u(-T) + \cdots \\ &= 1 + e^a + e^{2a} + \cdots + e^{na} \\ &= \sum_{k=0}^{n} e^{ka} \end{aligned}$$

$$y(nT) = \begin{cases} \dfrac{1 - e^{(n+1)a}}{1 - e^a}, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

예제 3.5 인과시스템의 임펄스응답 $h(nT)$와 입력수열 $x(nT)$가

$$h(nT) = e^{n\alpha}$$

$$x(nT) = \begin{cases} 1, & 0 \le n \le 4 \\ 0, & \text{기타} \end{cases}$$

와 같이 주어질 때, 응답 $y(nT)$를 구하라.

$$x(nT) = u(nT) - u(nT - 5T)$$

$$\begin{aligned} y(nT) &= \sum_{k=0}^{\infty} e^{k\alpha} u(nT - kT) - \sum_{k=0}^{\infty} e^{k\alpha} u(nT - 5T - kT) \\ &= \sum_{k=0}^{n} e^{k\alpha} - \sum_{k=0}^{n-5} e^{k\alpha} \end{aligned}$$

$$y(nT) = \begin{cases} \dfrac{1 - e^{(n+1)\alpha}}{1 - e^{\alpha}} & 0 \le n \le 4 \\[2ex] \dfrac{e^{(n-4)\alpha} - e^{(n+1)\alpha}}{1 - e^{\alpha}}, & n \ge 5 \end{cases}$$